Indian Statistical Institute, Bangalore

B. Math. (Hons.) Third Year

Second Semester - Differential Geometry II

Midterm Exam — Date: March 02, 2018

Maximum marks: 40 — Duration: 3 hours

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

1. let $A = \{(x_1, x_2, ...x_n) \in \mathbb{S}^{n-1}/x_1 \geq 0\}$ be the closed right half sphere.
   (a) Use the stereographic projection, show that $A \setminus \{(0, 0, ...1)\}$ is diffeomorphic to the closed right-half space $\{y \in \mathbb{R}^{n-1} \ \times \ 0/y_1 \geq 0\}$. [3]
   (b) Given any $P \in \partial\mathbb{D}^{n-1}$, find a diffeomorphism $\partial : \mathbb{D}^{n-1} \to A$ such that $\partial(P) = (0, 0, ...1)$. [3]
   (c) Conclude that if P is any point on the boundary of a closed disc $\mathbb{D}^{n-1}$, then $\mathbb{D}^{n-1} \setminus \{P\}$ is diffeomorphism to the closed upper-half space H^{n-1}. [2]
   (d) Identify $\mathbb{C}$ with $\mathbb{R}^2$ by the formula $z = x + iy$. Find the matrix of the derivative $df(z)$ for $f : \mathbb{C} \to \mathbb{C}$, where $f(z) = z^n$, in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. [2]

2. (a) Recall that we have charts on $\mathbb{RP}^2$ given by
   $$[x, y, z] \to (u_1, u_2) = (x/z,\ y/z) \text{ on } U_3 = \{z \neq 0\},$$
   $$[x, y, z] \to (v_1, v_2) = (x/y,\ z/y) \text{ on } U_2 = \{y \neq 0\},$$
   $$[x, y, z] \to (w_1, w_2) = (y/x,\ z/x) \text{ on } U_1 = \{x \neq 0\}.$$
   Show that there is a vector field on $\mathbb{RP}^2$ which in the last coordinate chart above has the coordinate expression $w_1\partial/\partial w_1 - w_2\partial/\partial w_2$. What are the expressions for this vector field in the other two charts? [2 + 2 + 2 = 6]
   (b) Consider the vector field on $U = (0, \infty) \subset \mathbb{R}$ given by $X(x) = \frac{1}{x}\frac{d}{dx}$. Find the local flow associated to X and the maximal intervals of existence. Check that the defining properties of a local flow are satisfied for this example. [2 + 2 + 2 = 6]
   (c) Let $X \in \chi(\mathbb{R})$ be the vector field $X = e^t\frac{d}{dt}$.

i. For $a \in \mathbb{R}$, compute the integral curve $\gamma_a : (-\epsilon_a, \delta_a) \to \mathbb{R}$ to X through a. (Be sure to specify its domain).

ii. Find the flow determined by X. Is X a complete vector field? [2 + 2 = 4]

3. (a) On $\mathbb{R}^3$, let X, Y, Z be the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z},$$

$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Show that $aX + bY + cZ \to (a, b, c) \in \mathbb{R}^3$ is a Lie algebra isomorphism (from a certain set of vector fields to $\mathbb{R}^3$) and that $[U, V] \to$the cross-product of the images of U and V. [4]

(b) Give an example of a distribution which is not integrable. [2]

4. (a) Show that $GL(n, \mathbb{H})$ is path connected. [2]

(b) Prove that $\det(A) > 0$ for all $A \in GL(n, \mathbb{H})$. [2]

(c) Show that the group $\mathbb{S}^3/\{-1, +1\}$ is isomorphic to $SO(3)$. [4]